

Fidelity decay of the two-level bosonic embedded ensembles of Random Matrices

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Abstract. We study the fidelity decay of the k -body embedded ensembles of random matrices for bosons distributed over two single-particle states. Fidelity is defined in terms of a reference Hamiltonian, which is a purely diagonal matrix consisting of a fixed one-body term and includes the diagonal of the perturbing k -body embedded ensemble matrix, and the perturbed Hamiltonian which includes the residual off-diagonal elements of the k -body interaction. This choice mimics the typical mean-field basis used in many calculations. We study separately the cases $k = 2$ and 3 . We compute the ensemble-averaged fidelity decay as well as the fidelity of typical members with respect to an initial random state. Average fidelity displays a revival at the Heisenberg time, $t = t_H = 1$, and a freeze in the fidelity decay, during which periodic revivals of period t_H are observed. We obtain the relevant scaling properties with respect to the number of bosons and the strength of the perturbation. For certain members of the ensemble, we find that the period of the revivals during the freeze of fidelity occurs at fractional times of t_H . These fractional periodic revivals are related to the dominance of specific k -body terms in the perturbation.

Keywords: Fidelity; Random Matrix Theory; k -body Embedded Ensembles; Boson Systems

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1. INTRODUCTION

Fidelity, also named quantum Loschmidt echo, is a measure of the sensitivity of the dynamics of quantum systems to perturbations and has attracted a lot of attention in the last years in connection with quantum information processes and quantum chaology; for a recent review see [1].

Fidelity compares the time evolution of a given initial state under a reference Hamiltonian with the time evolution of the same state under a slightly different one. We denote as \hat{H}_0 the reference or unperturbed Hamiltonian, and as $\hat{H}_\varepsilon = \hat{H}_0 + \varepsilon \hat{V}$ the perturbed one which includes the residual interaction. The former is associated with the unitary time-evolution $\mathcal{U}_0(t) = \hat{T} \exp[-\frac{i}{\hbar} \int_0^t d\tau \hat{H}_0(\tau)]$, where \hat{T} is the time-ordering operator; likewise, $\mathcal{U}_\varepsilon(t)$ is the propagator associated with the perturbed Hamiltonian. In the definitions of the propagators we have considered the most general case in which the Hamiltonians may depend on t explicitly. With these definitions, and taking an arbitrary initial state $|\Psi_0\rangle$, the fidelity amplitude is defined as

$$f_\varepsilon(t) = \langle \Psi_0 | \mathcal{U}_0(-t) \mathcal{U}_\varepsilon(t) | \Psi_0 \rangle, \quad (1)$$

whose square modulus is known as the *fidelity*

$$F_\varepsilon(t) = |f_\varepsilon(t)|^2. \quad (2)$$

Fidelity decay can occur in many different ways and in successive combinations of these. For an extremely short time, the so-called Zeno time, all systems will decay quadratically in time, both for chaotic and integrable systems. After the Zeno time systems specific correlations often dominate the decay.

For regular systems the decay typically changes into a quadratic and later Gaussian decay dominated by the dimension of the effective Hilbert space that is accessible to the wave function. Furthermore revivals are very common, particularly if the system has few degrees of freedom.

For chaotic systems on the other hand we expect, after the system specific behavior, linear or exponential decay of the fidelity which is determined by the coupling strength. This phase is often known as Fermi Golden Rule decay. Around the Heisenberg time the exponential decay changes over into a quadratic function or a Gaussian. An alternative behavior, known as Lyapunov decay, can occur with rather strong perturbations. For times of the order of the Ehrenfest time, exponential decay can be observed. The Lyapunov exponent, and not the perturbation strength, define the decay rate.

An alternative and interesting behavior is found if the diagonal part of the perturbation (or at least its time-average) vanish in the basis in which the unperturbed Hamiltonian is diagonal. Then we get a fidelity freeze, which has been shown for classically integrable [3] and chaotic [4] systems, as well as for random matrix models [5]. Both, the value at which the freeze occurs and the length of its duration, are functions of perturbation strength. For fermions, it was proposed that a mean field Hamiltonian as the reference system and the residual interaction could well lead to a fidelity freeze [6]. Specifically, this problem was treated in the framework of an embedded two-body random ensemble for fermions [7]. While there is no freeze on average, because of long tails in the distribution, typical samples from the ensemble display the freeze and indeed the median does [6].

In this paper we study the fidelity decay for the k -body embedded ensembles of random matrices for bosons, which are distributed over two single-particle states. We shall consider a reference Hamiltonian which consists purely of a diagonal 1-body terms as well as the diagonal k -body matrix elements, with $k = 2$ or $k = 3$, which are the physical important cases. The perturbed Hamiltonian will include a traceless residual interaction, consisting of the off-diagonal matrix elements of the same k -body interaction. Our results indicate that ensemble-averaged fidelity decays quadratically on time for short times (before the Heisenberg time t_H) as well as for very long times, precisely after the freeze of fidelity ends. Furthermore, freeze of fidelity is observed for the ensemble-averaged fidelity, and during the freeze, it displays periodic revivals with the periodicity given by the Heisenberg time. These results occur *often but not always*, when individual members of the ensemble are considered. We present numerical results showing that the periodicity of the revivals is an integer fraction of t_H , and demonstrate that the specific value of this period is related with specific off-diagonal terms of the k -body interaction considered in the perturbation.

2. THE k -BODY TWO-LEVEL BOSONIC ENSEMBLE OF RANDOM MATRICES

The k -body Embedded Ensemble of Random Matrices for bosons considers all possible k -body interactions among spin-less n -boson states, where the bosons are distributed over l single-particle states [8]. Below we discuss the simplest case $l = 2$. To define this ensemble, we first introduce the single-particle states associated with the operators \hat{a}_j^\dagger and \hat{a}_j , with $j = 1, 2$ ($l = 2$), which, respectively, create or annihilate one boson on the single-particle level j . These operators satisfy the usual commutation relations for bosons, i.e., $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$. The normalized n -boson states are denoted as $|\mu_r^{(n)}\rangle = (\mathcal{N}_r^{(n)})^{-1} (\hat{a}_1^\dagger)^r (\hat{a}_2^\dagger)^{n-r} |0\rangle$, where $\mathcal{N}_r^{(n)} = [r!(n-r)!]^{1/2}$ is a normalization constant and $|0\rangle$ is the vacuum state. The Hilbert-space dimension is thus $N = n + 1$. Then, in second-quantized form, the most general k -body interaction of n bosons distributed over two single-particle levels, $\hat{H}_k^{(\beta)}$, can be written as [9]

$$\hat{H}_k^{(\beta)} = \sum_{r,s=0}^k v_{r,s}^{(\beta)} \frac{(\hat{a}_1^\dagger)^r (\hat{a}_2^\dagger)^{k-r} (\hat{a}_1)^s (\hat{a}_2)^{k-s}}{\mathcal{N}_r^{(k)} \mathcal{N}_s^{(k)}}. \quad (3)$$

Here, k denotes the rank of the interaction, $1 \leq k \leq n$, and $v_{r,s}^{(\beta)}$ are the k -body matrix elements, which are independent Gaussian-distributed random numbers with zero mean and constant (fixed) variance $v_0^2 = 1$. As in the case of the canonical random matrix ensembles (see [10]), Dyson's parameter β distinguishes the cases according to time-reversal invariance: $\beta = 1$ is the case where time-reversal invariance holds, and $\beta = 2$ where this symmetry is broken. Hence, the k -body interaction matrix $v^{(\beta)}$ is a member of the Gaussian orthogonal ensemble (GOE) for $\beta = 1$, or Gaussian unitary ensemble (GUE) for $\beta = 2$. Note that $\hat{H}_k^{(\beta)}$ commutes with the number operator $\hat{n} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2$, i.e., the interaction preserves the total number of bosons n .

This ensemble presents some noteworthy properties: It exhibits non-ergodic level statistics, i.e., spectral or ensemble unfolding do not yield the same results [8]. In addition to this, for $\beta = 1$ the ensemble displays a large and robust quasi-degenerate portion of the spectrum for a wide interval of k , while for $\beta = 2$ only seldom accidental quasi-degeneracies are observed [11]. These results are due to the fact that each member of the ensemble is Liouville integrable in the semiclassical limit, for all values of k ; see Ref. [12] for details.

3. NUMERICAL RESULTS ON FIDELITY DECAY

3.1. Definitions

In order to study fidelity, as stated above, we must first define the reference (or unperturbed) and the perturbed Hamiltonians considered. The reference Hamiltonian $\mathcal{H}_{0,k}$ shall be defined as the sum of a fixed one-body interaction term, and the diagonal part of the k -body interaction, with either $k = 2$ or $k = 3$. Since each of these terms

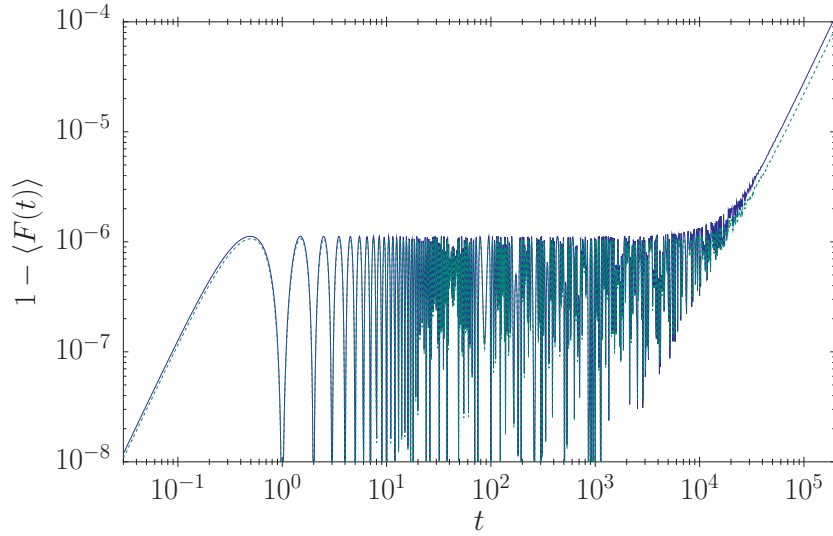


FIGURE 1. Comparisson between the time evolution of the ensemble-averaged fidelity illustrated with $1 - \langle F(t) \rangle$ when the diagonal part of the k -body interaction, which is included in the reference Hamiltonian $\hat{\mathcal{H}}_{0,k}$, is fixed (upper curve for long times) or varied with the realizations (lower curve for long times) for different realizations of the ensemble.

have different spectral widths [8], we normalize each one with the respective width of the spectrum W_k . This mean-field unperturbed Hamiltonian is further restricted by imposing that $\hat{\mathcal{H}}_{k,0}$ is diagonal in the occupation number basis, which we shall use as our reference basis. Notice that the reference Hamiltonian contains the diagonal part of the k -body interaction, i.e., it includes the k -body matrix elements of the form $v_{r,r}^{(\beta)}$. Then, the unperturbed Hamiltonian is explicitly given by

$$\hat{\mathcal{H}}_{0,k} = \frac{1}{W_{k=1}} \hat{H}_{k=1}^{\text{diag}} + \frac{\lambda}{W_k} \hat{H}_k^{\text{diag}}. \quad (4)$$

In Eq. (4), the width of the spectrum of the k -body embedded ensemble is given by

$$W_k = \frac{1}{N} \overline{\text{tr}[H_k^{(\beta)}]^2} = \Lambda_B^{(0)}(k) + \frac{\delta_{\beta 1}}{N} \sum_{s=0}^k \Lambda_B^{(s)}(n-k), \quad (5)$$

where

$$\Lambda_B^{(s)}(k) = \binom{n-s}{k} \binom{n+s+1}{k}. \quad (6)$$

In these expressions we have used explicitly the restriction to the two single-particle level case ($l = 2$); the label B stands for bosons, the over-line indicates ensemble average, and $\Lambda_B^{(s)}(k)$ is the s -th eigenvalue of the ensemble-averaged correlation matrix of the bosonic k -body embedded ensemble; see Ref. [8] for details.

As for the residual interaction, it simply consists of the remaining off-diagonal matrix elements of the k -body interaction, properly normalized by W_k . Denoting by λ the

perturbation strength, the perturbed Hamiltonian $\hat{\mathcal{H}}_{\lambda_k}$ is written as

$$\hat{\mathcal{H}}_{\lambda_k} = \hat{\mathcal{H}}_{0_k} + \frac{\lambda}{W_k} \hat{H}_k^{\text{offdiag}}. \quad (7)$$

Notice that with this definition of the reference Hamiltonian and the traceless residual interaction, where $\hat{\mathcal{H}}_{0_k}$ also depends upon λ , the conditions to observe fidelity freeze are fulfilled [5].

3.2. Ensemble-averaged fidelity decay

Here, we present numerical results for the ensemble-averaged fidelity decay in terms of the perturbation strength λ and the number of particles n . For the numerical treatment, we have fixed the diagonal part of the k -body perturbation to a specific realization of the ensemble. Therefore, the different realizations of the ensemble will only involve different realizations of the off-diagonal part of the perturbation. This was done for numerical convenience. We emphasize that this implementation has no effect in the results; this is shown in Fig 1, where the two curves included are difficult to distinguish quantitatively. In order to obtain the physically relevant scalings, it is convenient to rescale the physical time t' by a dimensionless time measured in units of the Heisenberg time, i.e., $t = t'/t_H$. Here, $t_H = 2\pi\hbar/d$ and d denotes the average level-spacing of the spectrum of the unperturbed Hamiltonian $\hat{\mathcal{H}}_{0_k}$.

The results for ensemble average were obtained as follows: We first fixed the unperturbed Hamiltonian $\hat{\mathcal{H}}_{0_k}$, and considered 1000 independent realizations of the perturbation (involving only the off-diagonal part of the k -body interaction); for a given perturbation strength λ and fixing the number of bosons n , this defined the perturbed Hamiltonian $\hat{\mathcal{H}}_{\lambda_k}$ through Eq. (7). For each realization of the residual interaction $\hat{H}_k^{\text{offdiag}}$, we calculated the corresponding time evolutions from an initial random state, which permitted to obtain the fidelity for this particular realization, using Eq. (2). Averaging the resulting fidelities over different realizations constitutes the ensemble-averaged fidelity.

In Fig. 2 we present numerical results for the $1 - \langle F(t) \rangle$ as a function of time (in Heisenberg time units), as convenient representation of ensemble-averaged fidelity decay, for various values of the perturbation strength λ (for $n = 1000$), and various values of the number of particles n (for $\lambda = 10^{-6}$). The cases illustrated corresponds to $k = 2$ and $\beta = 1$; similar results were obtained for $k = 3$ as well as when the time-reversal symmetry does not hold ($\beta = 2$). As shown in Fig. 2, for times smaller than the Heisenberg time $t < t_H = 1$, but longer than Ehrenfest time, fidelity displays a quadratic decay in time, which is the typical situation observed for integrable systems. At the Heisenberg time ($t = t_H = 1$), the system exhibits a revival, i.e., $\langle F(t) \rangle$ approaches the unity again. This revival is natural for integrable 1d systems [1], which corresponds to this case, but is also observed for the Gaussian ensembles of Random Matrices [13], or in models with underlying chaotic dynamics [14].

Beyond the Heisenberg time, fidelity remains essentially constant at certain value F_{plateau} ; this is the fidelity freeze. The value F_{plateau} scales as λ^2 and n^2 with respect to the perturbation strength and the number of particles, respectively. We note that during

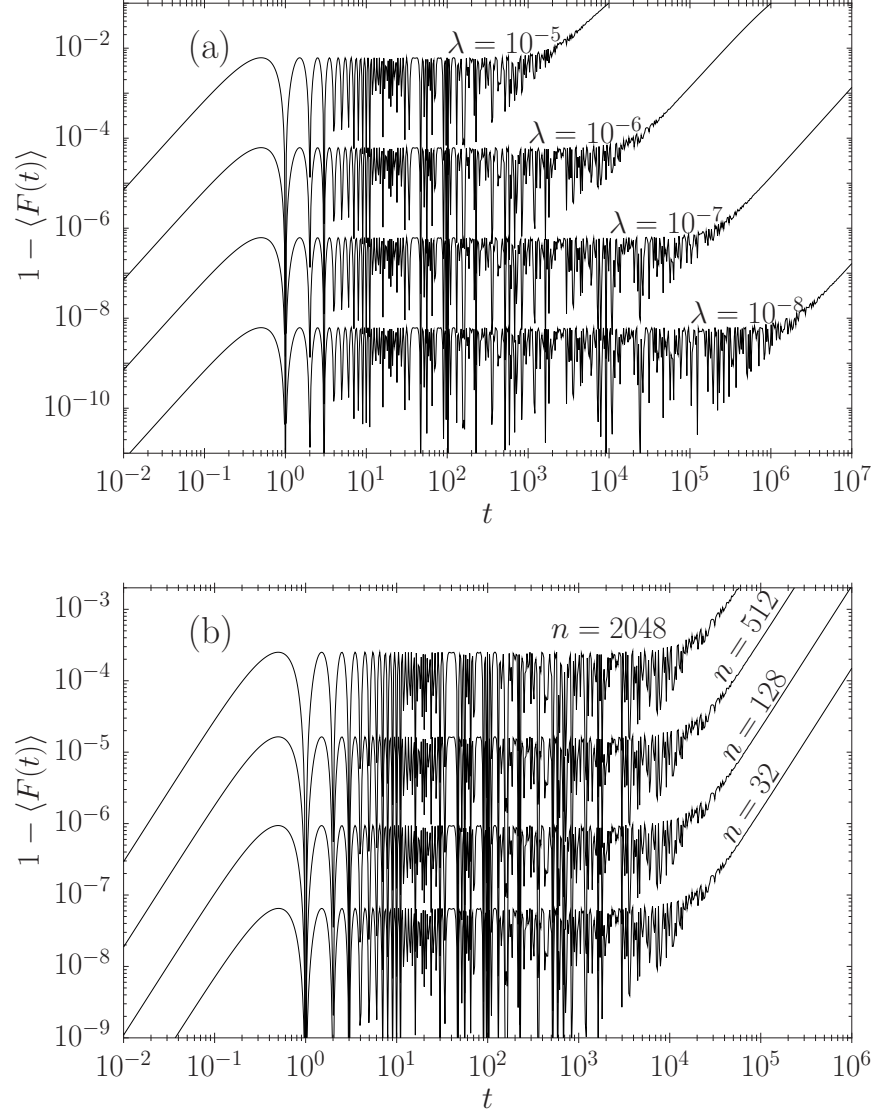


FIGURE 2. Time evolution of $1 - \langle F(t) \rangle$ for $k = 2$, $\beta = 1$. (a) Dependence upon the strength of the perturbation ($n = 1000$ particles); (b) dependence upon the number of particles for $\lambda = 1 \times 10^{-6}$. For very short and very long times, the fidelity $\langle F(t) \rangle$ scales as t^2 . (c) Illustration of the periodic revivals displayed by the ensemble-averaged fidelity during the freeze. Note that the periodicity is the Heisenberg time.

the freeze, fidelity displays some very short periodic revivals, whose period is precisely the Heisenberg time $t_H = 1$. To the best of our knowledge, such periodic revivals during the freeze of fidelity have not been observed. The freeze of fidelity lasts until some ending time t_e is reached, which displays a λ^{-1} dependence. After t_e , fidelity decays once again quadratically in time, until it saturates near $F = 0$.

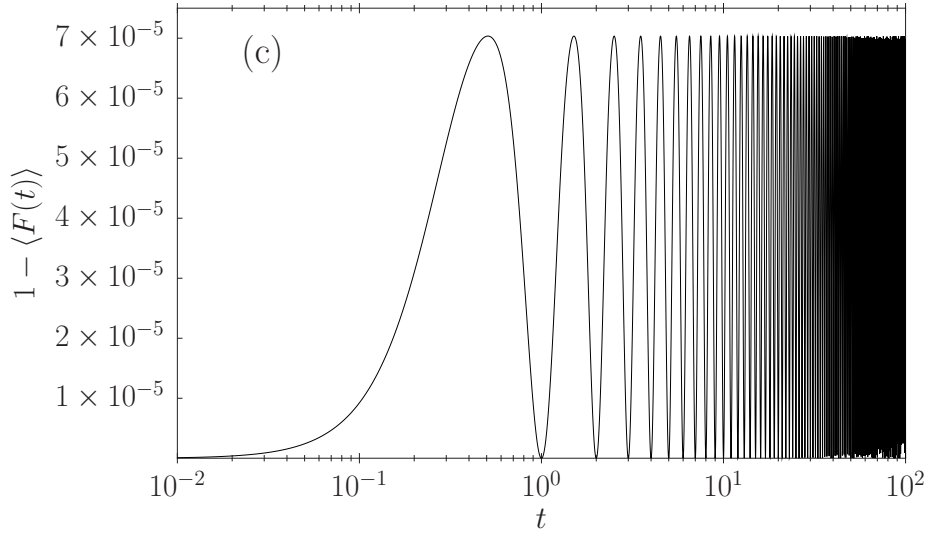


FIGURE 2. (Continued)

3.3. Fidelity decay of specific members of the ensemble

Fidelity, for most of the realizations of the ensemble, resembles qualitatively the ensemble-average fidelity. Yet, for some specific realizations the results do differ. A common case is that the position of the plateau may be slightly shifted up or down with respect to the average one.

A more interesting case occurs for some specific realizations of the ensemble, where the differences are more subtle and far reaching; see Figs. 3 and 4. Indeed, for some specific members of the ensemble, fidelity displays periodic revivals during the freeze with an integer fractional period. That is, it displays a fractional period with respect to the unit, which is the Heisenberg time. This period has the form $T = 1/c$, with c an integer whose specific value is linked with the value of the rank of the interaction k of the residual interaction.

These fractional periodic revivals are observed when some specific matrix elements of the k -body interaction matrix $v_{r,s}$ dominate over the rest. In fact, these matrix elements correspond to eliminating precisely c particles from one single-particle state, and creating them in the other one. By definition of the k -body interaction, c is bounded as $1 \leq c \leq k$. Therefore, for $k = 2$ we may move at most two particles from one single-particle state to the other one. When the associated matrix elements somehow dominate the perturbation, fractional periodic revivals are observed with period $1/2$ in units of t_H (see Fig. 3). Likewise, for $k = 3$ there are perturbing terms that involve moving up to one, two or three particles. The relative dominance of these terms is related to the observation of periodic revivals with periods 1 , $1/2$ or $1/3$ in Heisenberg-time units, respectively (see Fig. 4). We note that these results do not depend on the time-reversal invariance of the ensemble, given by β .

The results discussed above are illustrated in Figs. 3 and 4. In Fig. 3 we present the fidelity decay for short times (in units of t_H) of two realizations of the ensemble

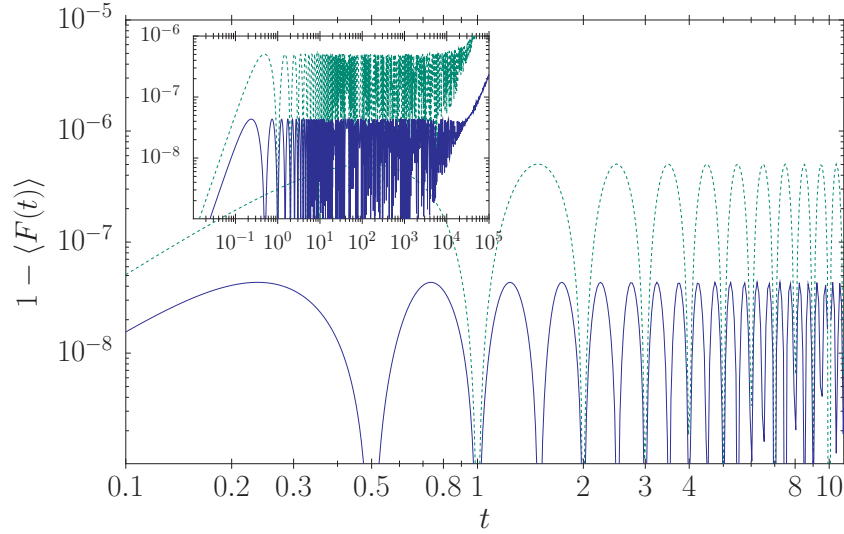


FIGURE 3. Fidelity decay for small times of two specific realizations of the ensemble for $k = 2$ ($n = 128$ and $\lambda = 10^{-6}$). During the freeze of fidelity, periodic revivals are displayed, with period 1 (dotted curve), and of period $1/2$ (solid curve), in units of the Heisenberg time. The inset presents the same results for longer times.

for $k = 2$ ($n = 128$ and $\lambda = 10^{-6}$). The dotted curve displays a case comparable with the ensemble-average fidelity, where the terms moving one particle dominate the perturbation; the solid curve illustrates a case where the oscillations of period $1/2$ appear. In the latter case, the dominating two-body interaction $\hat{H}_k^{\text{offdiag}}$ corresponds to the term $(\hat{a}_1^\dagger)^2(\hat{a}_2)^2 + (\hat{a}_2^\dagger)^2(\hat{a}_1)^2$. Figure 4 illustrates the $k = 3$ case. As discussed above, in the case of three-body interactions it is possible to observe oscillations of period 1 (dashed line), $1/2$ (dotted line) or $1/3$ (solid line), depending on the specific k -body matrix elements that dominate the interaction.

4. CONCLUSIONS

In this paper we have presented numerical results on the fidelity decay of the two-level bosonic k -body embedded ensembles of random matrices, for $k = 2$ and $k = 3$, considering the ensemble-averaged fidelity and comparing it with some individual realizations of the ensemble. The reference Hamiltonian is diagonal in the occupation number basis, where the residual interaction is traceless. We have obtained the relevant scaling laws, and observed the fidelity freeze, which displays oscillations of period 1 in units of the Heisenberg time.

We have also presented preliminary numerical results on the existence of fractional periodic revivals (in units of the Heisenberg time) during the freeze of fidelity, for some specific members of this ensemble. Expressed in units of the Heisenberg time, the period of these revivals is $T = 1/c$, where c is an integer number related to the specific k -body interactions which dominate the perturbation. The occurrence of these fractional periodic revivals is somewhat rare with respect to ensemble average; we conjecture

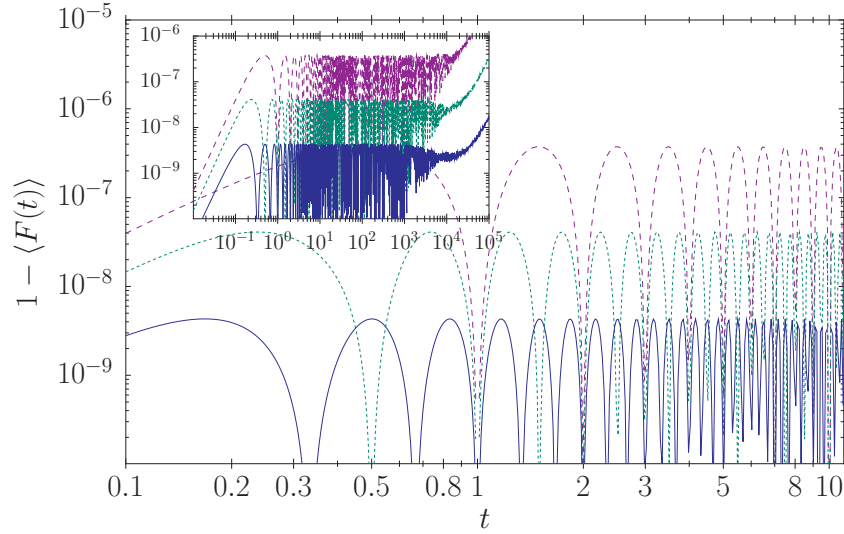


FIGURE 4. Same as Fig. 3 for $k = 3$ ($n = 128$ and $\lambda = 10^{-6}$), displaying cases where the periodicity of the revivals is 1 (dashed curve), $1/2$ (dotted curve) or $1/3$ (solid curve), in units of the Heisenberg time. The inset shows the same results for longer times.

that this is related to the relative number of terms in the perturbation which move c particles from one level to the other. Our results indicate that the appearance of the fractional periodic revivals is independent of the time-reversal invariance. These results may be interesting for the understanding —and even the measurement— of three-body interactions in two-component Bose-Einstein condensates [15].

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